ERGODIC ASPECTS OF PARTIALLY HYPERBOLIC DYNAMICS

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Even though the theory of dynamical systems developed very rapidly during the last decades, at the present state of the art in many aspects one completely understands only low-dimensional (conformal) and uniformly hyperbolic dynamical systems such as, for example, hyperbolic surface diffeomorphisms. In particular, the thermodynamic formalism initiated in the 70s by Sinai, Ruelle, and Bowen, nowadays provides a powerful tool to describe the interplay of topological dynamics with the ergodic theory of measures which are invariant under the dynamics. The principle ingredients of this formalism are entropy, pressure, free energy, equilibrium states, Lyapunov exponents, and Birkhoff averages. It is also intimately related with the theory of fractal dimensions.

The aim of this mini-course is to study higher-dimensional and nonhyperbolic dynamical systems with special emphasis on skew-products and partially hyperbolic diffeomorphisms. In these cases, especially when the fiber dynamics or the central nonhyerbolic direction is one-dimensional we will discuss some recent progress in the theory.

One expects that the non-hyperbolicity of the systems must be reflected in the existence of non-hyperbolic invariant ergodic measures (i.e., having some zero Lyapunov exponents). We study this question, especially in the setting of the so-called robustly non-hyperbolic transitive diffeomorphisms. We will present first a method based on periodic approximations introduced in [14] in the setting of skew-products, and later used to construct open sets of diffeomorphisms with such measures [16] and applied to generic non-hyperbolic homoclinic classes of diffeomorphisms [12, 5]. This approach provides conditions for a sequence of atomic measures to converge to a nontrivial nonhyperbolic ergodic measure. More recently, [1] introduced a method based on the so-called blenders and flip-flop for constructing nonhyperbolic ergodic measures with positive entropy. We will review these constructions and explain their main ingredients. If the time allows, we also discuss some variations of these methods and some perspectives ([2, 3, 6]). Let us emphasise that in the setting of thermodynamical formalism of nonhyperbolic systems nonhyperbolic ergodic measures with positive entropy play an essential role that can not be discarded.

To develop this formalism and in order to understand higher-dimensional and nonhyperbolic dynamical systems, one natural approach is to consider classes of systems with gradually increasing type of complexity. Hence, it is very natural to investigate diffeomorphisms which have a (step) skew-product structure with a low-dimensional surface diffeomorphism as base map and one-dimensional fiber maps.

Motivated by this, we study a family of partially hyperbolic diffeomorphisms of step skew-product type modeled over a horseshoe with interval or circle fiber maps. These systems are genuinely nonhyperbolic containing intermingled horseshoes (and periodic points) of different hyperbolic behavior (contracting and expanding center).

The associated invariant set possesses a very rich topological fiber structure, it contains uncountably many trivial and uncountably many nontrivial fibers (see [7]).

Starting with a class of very simple paradigmatic models we study transitive step skew-products modelled over a shift with one-dimensional fiber maps. In case the fiber maps have both contracting and expanding regions, this dynamics is genuinely nonhyperbolic and simultaneously exhibits ergodic measures with positive, negative, and zero exponents. Moreover, it contains a gap (at least one) which is associated with (in fact caused by) an "exposed" ergodic measure, [8, 7].

We introduce a set of axioms to capture the essential dynamic features and analyze its topological dynamics. It turns out that those axioms equivalently characterize nonhyperbolic robustly transitive maps. We also study ergodic and thermodynamic properties by analyzing the space of ergodic hyperbolic (with either expanding or contracting fiber exponent) and nonhyperbolic measures in the weak$*$ topology and in entropy. Our methods include the explicit construction of hyperbolic sets based on an approximation using so-called skeletons, multi-variable-time horseshoes, and our set of axioms, [11, 9]. Finally, we also explain how these constructions and ideas can be translated to the setting of partially hyperbolic diffeomorphisms, [9, 10].

Our results have applications to the description of the space of ergodic measures in the two settings discussed above (see for further developments [4, 15, 13, 6].

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Schedule of the course

- Nonhyperbolic step skew-products – Porcupine-like dynamics
- Nonhyperbolic step skew-products – An axiomatic approach
- Construction of non-hyperbolic ergodic measures – Method of periodic approximation
- Approximation of non-hyperbolic ergodic measures – Katok's construction and skeletons
- Construction of non-hyperbolic ergodic measures – Blenders, flip-flops, and positive entropy
- Approximation of nonhyperbolic ergodic measures in a partially hyperbolic context

Talks

Convex sum of hyperbolic measures (Katrin Gelfert)

Abstract: In the uniformly hyperbolic setting it is well known that the measure supported on periodic orbits is dense in the convex space of all invariant measure. We study the reverse question: assuming that some ergodic measure converges to a convex sum of hyperbolic ergodic measures, what can be deduced on the initial measures? To every hyperbolic measure whose stable/unstable Oseledets splitting is dominated we associate canonically a unique class of periodic orbits for the homoclinic relation, that we call its intersection class. In a dominated setting, we prove that a convex sum of finitely many ergodic hyperbolic measures of the same index is accumulated by ergodic measures if, and only if, they share the same intersection class. We provide examples which indicate the importance of the domination assumption. This is joint work with Christian Bonatti and Jairo Bochi.

About the space of ergodic measures in partially hyperbolic systems (Lorenzo J. Díaz)

Abstract: We will study transitive sets (typically, homoclinic classes) which are partially hyperbolic with one dimensional center direction. We are specially interested in the case where this direction is genuinely non-hyperbolic (i.e., there are some hyperbolic periodic points which are expanding in the central direction and other periodic points which are contracting).

In this setting, the space of ergodic measures splits into three parts according to the exponent corresponding to the central direction: positive (expanding), negative (contracting), and zero (neutral). In many cases, in very rough terms, the expanding and contracting measures are glued throughout the neutral ones. But this is not always the case, and in some case special configurations arise. A key ingredient in those discussions are the so-called exposed pieces of dynamics.